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Composite boson dominance in many-fermion systems

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Abstract. I recently proposed a method of bosonization based on the use of coherent states of fermion composites, whose validity was restricted to smooth structure functions. In the present paper I remove this limitation and derive results which hold for arbitrary interactions and structure functions. The method respects all symmetries and in particular fermion number conservation. It reproduces exactly the results of the pairing model of atomic nuclei and of the BCS model of superconductivity in the number conserving form of the quasi-chemical equilibrium theory.

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1 Introduction

There are many finite and infinite fermion systems whose partition function at low energy is dominated by bosonic modes. This is always the case when, due to spontaneous breaking of a global symmetry, there are Goldstone bosons. The effective bosons of these systems can be charged (fermion number 2), like Cooper pairs in metal, nuclear and so called color superconductivity [1] or neutral (fermion number 0), like phonons in condensed matter and pions in hadronic physics.

The structure of composite bosons can also be changed by varying temperature or control parameters: for instance strongly interacting fermionic atoms in magnetic traps can form at some temperature molecular pairs which however condense only at a lower temperature. Moreover at zero temperature molecular pairs can be transformed into Cooper pairs by tuning an external magnetic field which controls a Feshbach resonance. For such systems two features are most relevant: the role of molecular versus Cooper pairs and the contribution of non condensed pairs to the ground state energy [2].

There is an immense literature concerning techniques to derive effective Hamiltonians for both types of effective bosons. I will refer to all of them as bosonization methods. To put my work into perspective I will briefly review the most relevant step in the development of this subject, but with no attempt to completeness.

The first approach to bosonization of which I am aware is due to Bogoliubov [3]. After the BCS work on superconductivity Bogoliubov reformulated their theory using the Fröhlich Hamiltonian [4] of electrons interacting with lattice phonons and rederived all their results concerning ground state properties. He then mapped the Cooper pairs into effective bosons and found that their dynamics is described by the Hamiltonian of a superfluid system of elementary bosons he studied previously, whose excitations are called Anderson-Bogoliubov sound. These modes are related to a nonvanishing contribution of non condensed pairs to the ground state energy density.

The presence of bosons external to the electron system, the lattice phonons, plays an essential role in Bogoliubov's theory. But in atomic nuclei in which nucleons are assumed to interact via a nucleon-nucleon potential (without mesons), in gaseous systems of fermionic atoms and in the BCS model there are no external bosons. In particular in the BCS model the contribution of noncondensed pairs to the ground state energy density must vanish since the BCS solution for the ground state energy-density is exact in the thermodynamic limit [5], and therefore collective excitations cannot coincide with the Anderson-Bogoliubov sound.

The importance of superconductivity in atomic nuclei was immediately understood by Bohr Mottelson and Pines [6] and the method of BCS, which breaks fermion number conservation was adapted to atomic nuclei [7] for which this symmetry is important. Immediately superconductivity of infinite nuclear matter was investigated by many authors [8], and much later it was suggested that because of the strong tensor force Cooper pairs in symmetric nuclear matter should have the deuteron quantum numbers [9]. Subsequently a smooth transition from Bose-Einstein condensation of deuteron-like bound states at low densities and temperatures to BCS pairing at higher densities was considered [10].

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More recently the superconducting properties of ultrasmall metallic grains have been investigated [11]. In this context, like for atomic nuclei, it is important a theoretical method which respects fermion number conservation [12].

There have been many attempts to reformulate the nuclear Hamiltonian in terms of effective bosons. Beliaev and Zelewinski [13] made an organic theory of bosonization with some points common to the present method, but their expansion has problems of convergence and violates fermion number conservation. Later on much in the spirit of BCS but in a phenomenological approach Arima and Iachello [14] introduced two different composite bosons, the s- (spin 0) and d- (spin 2) bosons. Their model, the Interacting Boson Model, proved extremely successful in reproducing low energy nuclear properties, but it has not been derived in a fully satisfactory way from a fundamental nuclear Hamiltonian. One of the features of the model to be understood is why can one (mostly) restrict the boson space to the s- and d-bosons. A justification will emerge in the method I will present which, even though is found with a very specific fermion-fermion interaction, might have a more general validity. I must notice that the absence in the Interacting Boson Model of bosons with the deuteron quantum numbers predicted in infinite nuclear matter might be due to the large size [9] of the deuteron-like Cooper pairs (very roughly) estimated of about 17 fm, and the fact that in heavy atomic nuclei the valence shells of the protons are different from those of neutrons.

In general there are many methods of bosonization. In 1+1 dimensions there is a wealth of exact results [15]. These results have been extended [16] to many dimensions: first by use of renormalization group transformations the fermion space is reduced to a tiny shell around the Fermi surface, then the problem is essentially reduced to one dimension by considering only excitations normal to the Fermi surface. In this way many general properties can be investigated, but in many cases the effective parameters introduced by renormalization group transformations are difficult to evaluate.

Different approach in multidimensional problems are based on several recipes [17] for mapping of a fermion model space into a boson space. Such methods respect fermion number conservation and in principle yield an exact solution to the problem, but in practice one has to perform a truncation in the fermion space related to a selection of degrees of freedom guided by physical insight and calculational convenience. One shortcoming of this procedure is the appearance of "intruders", namely states which in spite of their low energy do not appear in the boson space generated by the mapping [18].

For many-body systems like fermionic gases in magnetic traps the most common approaches are based on numerical simulations or the quasi-chemical equilibrium theory [19,20], but a true bosonization to my knowledge has not been achieved.

I developed an approach to bosonization [21] in which an effective bosonic action is derived by evaluating the fermion partition function in a basis of coherent states of fermionic composites. Coherent states offer for composites the same advantages they give for elementary bosons and fermions.

After bosonization is achieved the fermion dynamics can be studied by functional or numerical methods [22]. Analytic calculations are also possible in this approach both in the path integral and Hamiltonian formalisms, at the price of an expansion in the inverse of the dimension of the fermion space. Such an expansion respects fermion number conservation and therefore can be used also for finite systems, and in fact its first application was to atomic nuclei.

An important issue was left over in the original paper. It concerns a subtraction necessary to perform the expansion in the presence of a condensate. The expansion was done under the assumption that the structure function of the condensed bosons be almost constant, a restriction which can be fulfilled in some atomic nuclei but is not in most systems, including the BCS model. For this reason it was not possible to test the method on this theory. This limitation is removed in the present paper, and *general results are derived for arbitrary fermion-fermion interactions and structure functions of the composites.* Actually the formalism is more general, as it will be illustrated at the end of Section 3, where the effective bosonic action is reported. But let me anticipate that it has been applied also to relativistic field theories, including gauge theories, at zero [23] and finite fermion number [24]. In the nonrelativistic domain the present method of bosonization finds its potential applications in finite systems like atomic nuclei and small metallic grains and in systems with nonseparable interactions like BEC of fermion gases.

The paper is organized in the following way. To make it reasonably selfcontained I included some material from references [21,23]. In Sections 2 and 3 I outline the approach and the derivation of the effective bosonic action [21], whose details can be found in Appendix D. Section 4 contains the derivation, valid for arbitrary fermion-fermion interaction, of the effective boson Hamiltonian which in Section 5 is expanded in the inverse of the dimension of the fermion space. The leading order, given by equation (47), has the form of the Bogoliubov model of a superfluid boson system.

As a test of the method for finite systems I apply it in Section 6 to the pairing model of nucleons in a single j-level. *The spectrum [25] is correctly reproduced in a form useful for an understanding and justification of the Interacting Boson Model of nuclear physics.*

Section 7 contains an application to infinite systems. All the results of the BCS model of superconductivity concerning the ground state are exactly reproduced in the number conserving form of the quasi-chemical equilibrium theory [19].

In Section 8 I summarize the results and conclude with an outlook.

As already said in the present paper I do not deal with neutral composites of particle-hole type, but I will introduce some neutral composites constructed with charged composites. Particle-hole bosons will be discussed by an extension of the method in a future work.

At last I want to mention that the present approach has been applied also to relativistic field theories [23], including gauge theories. There are technical differences, due to ultraviolet divergencies in relativistic theories, which make necessary a different way of evaluating the effective action of the composite bosons. But the essential strategy remains the same based on the use of coherent states of composites in the framework of the transfer matrix formalism which is close to the Hamiltonian formalism of nonrelativistic theories. The method has been tested on models of fermions with large number of flavors and quartic interactions in 3+1 dimensions, exactly reproducing the gap equation for spontaneous breaking of a discrete chiral symmetry and the mass of the effective boson appearing in these models. Moreover the structure function of the condensed bosons has been determined for the first time. Its spatial part turns out, surprisingly enough, to be identical to that of the Cooper pairs of the BCS model.

2 Outline of the approach

Consider the partition function of a system of elementary (non composite) bosons

$$Z = \mathrm{tr}\left[\exp\left(-\frac{1}{T}(H_{elem} - \mu\,\hat{n})\right)\right] \tag{1}$$

where T is the temperature, μ the chemical potential and $\hat{n}$ the number operator. A sector of n particles can be selected by the constraint

$$T\frac{\partial}{\partial\mu}\ln Z = n. \tag{2}$$

A functional form of Z can be found by performing the trace over coherent states [26]

$$|\alpha\rangle = \exp\left(\sum_K \hat{\alpha}_K^\dagger \hat{\alpha}_K\right)|0\rangle, \tag{3}$$

where K are the particles quantum numbers, $\hat{\alpha}_K^\dagger$ their canonical creation operators and α_K holomorphic variables. Coherent states satisfy the basic or defining equations

$$\hat{\alpha}_K|\alpha\rangle = \alpha_K|\alpha\rangle \tag{4}$$

where $\hat{\alpha}_K$ are canonical destruction operators. In terms of these states we can write the identity in the Fock space

$$\mathcal{I} = \int d\mu(\alpha^*,\alpha)\langle\alpha|\alpha\rangle^{-1}|\alpha\rangle\langle\alpha| \tag{5}$$

where

$$d\mu(\alpha^*,\alpha) = \prod_K\left[\frac{d\alpha_K^* d\alpha_K}{2\pi i}\right]. \tag{6}$$

Using this resolution of the identity the trace in the partition function can be evaluated with the result [26]

$$Z = \int d\mu(\alpha^*,\alpha)\exp\left[-S_{elem}(\alpha^*,\alpha)\right]. \tag{7}$$

In the above equation S_{elem} is the action of the particles

$$S_{elem} = \tau\sum_t\{-\alpha_t^*\nabla_t\alpha_{t-1} + H_{elem}(\alpha_t^*,\alpha_{t-1}) - \mu\,\alpha_t^*\alpha_{t-1}\}, \tag{8}$$

where

$$\nabla_t f = \frac{1}{\tau}(f_{t+1} - f_t), \tag{9}$$

and $H_{elem}(\hat{a}^\dagger, \hat{a})$ is the Hamiltonian in normal order.

In a system of fermions whose low energy excitations are dominated by fermion composites I can restrict the trace to these composites. The restricted partition function can be written

$$Z_C = \mathrm{tr}\left[\mathcal{P}_C\exp\left(-\frac{1}{T}(H_F - \mu_F\,\hat{n}_F)\right)\right] \tag{10}$$

where $\mathcal{P}_C$ is a projection operator in the subspace of the composites. By analogy to elementary bosons I assume for it the approximate expression

$$\mathcal{P} = \int\frac{d\beta^*\,d\beta}{2\pi i}\langle\beta|\beta\rangle^{-1}|\beta\rangle\langle\beta| \tag{11}$$

where β^*, β are holomorfic variables and $|\beta\rangle$ coherent states of composites

$$|\beta\rangle = |\exp\Big(\sum_J \beta_J\hat{B}_J^\dagger\Big)\rangle. \tag{12}$$

The creation operators of composites with fermion number 2 are

$$\hat{B}_J^\dagger = \frac{1}{2\sqrt{\Omega_J}}\sum_{m_1,m_2}\hat{c}_{m_1}^\dagger\left(B_J^\dagger\right)_{m_1,m_2}\hat{c}_{m_2}^\dagger, \tag{13}$$

while those of fermion number zero are

$$\hat{\Phi}_J^\dagger = \frac{1}{\sqrt{\Omega_J}}\sum_{m_1 m_2}\hat{c}_{m_1}^\dagger\left(\Phi_J^\dagger\right)_{m_1 m_2}\hat{c}_{m_2}. \tag{14}$$

The $\hat{c}^\dagger$'s are fermion creation operators, m and J, represent all the fermion and boson quantum numbers and Ω_J is the index of nilpotency of the J-composite, which is defined as the largest integer such that

$$\left(\hat{B}_J\right)^{\Omega_J} \neq 0. \tag{15}$$

I will assume for simplicity the index of nilpotency equal to half the dimension of the fermion space for all the composites. The matrices B_J, Φ_J are *the structure functions of the composites, which must be determined by a variational calculation.* Only the solutions with high index of nilpotency are acceptable.

I call the states $|\beta\rangle$ coherent because they share with coherent states of elementary bosons the property of a fixed phase relation among the components with different number of composites. But the basic property of coherent states cannot be fulfilled. Indeed

$$\hat{B}_J|\beta\rangle \neq \beta_J|\beta\rangle. \tag{16}$$

This is a consequence of the composites commutation relations, which are not canonical

$$\left[\hat{B}_J, \hat{B}_K^\dagger\right] = \frac{1}{2}\mathrm{Tr}\,(B_J B_K^\dagger) - \hat{c}^\dagger B_K^\dagger B_J\,\hat{c}\,. \tag{17}$$

In states with a number of composites $n \ll \Omega$, the above equations can be approximately satisfied provided the structure functions are sufficiently smooth. Indeed in such a case the last term is of order n/Ω. But in states with $n \sim \Omega$, it is not possible to satisfy them even with an absolute freedom about the form of the structure functions (which are instead determined by the dynamics). The best we can do [21] is to satisfy them for states with $n+k$ composites, for fixed $n \sim \Omega$ and $|k| \ll \Omega$.

As stated in the Introduction, in the present work I will study only composites with fermion number 2. The properties of the operator $\mathcal{P}$ are reported in Appendix C in the form derived in reference [23].

The trace can be exactly evaluated [21] (see below) yielding a functional form of Z_C

$$Z_C = \int \left[\frac{d\beta^* d\beta}{2\pi i}\right] \exp\left(-S_{\mathit{eff}}(\beta^*, \beta)\right). \tag{18}$$

The expression of the effective bosonic action S_{eff} is reported in the next section. This result holds under the only physical assumption of boson dominance and the approximation adopted for $\mathcal{P}$.

In many-body physics it is often used the Hamiltonian formalism. The Hamiltonian of the effective bosons, H_B, cannot be read directly from the effective action, because $S_{\mathit{eff}}(\beta^*, \beta)$ does not have the form of an action of elementary bosons. Indeed it contains anomalous time derivative terms, anomalous couplings of the chemical potential and nonpolynomial interactions, which are all features of compositeness. Therefore it has been necessary to devise an appropriate procedure to derive H_B, which is given in terms of boson operators $\hat{b}^\dagger, \hat{b}$, (not to be confused with the composite operators $\hat{B}^\dagger, \hat{B}$) satisfying canonical commutation relations, so that

$$Z_C = \mathrm{tr}\left(-\frac{1}{T}(H_B - \mu_B \hat{n})\right). \tag{19}$$

μ_B is the boson chemical potential and $\hat{n}$ the boson number operator.

H_B has a closed form but, for a practical use, it is necessary to perform an expansion in inverse powers of the index of nilpotency Ω.

3 The effective bosonic action

The most general fermion-fermion interaction can be written as a sum of separable terms, so that the fermion Hamiltonian can be given the form

$$H_F = \hat{c}^\dagger h_0\,\hat{c} - \sum_K g_K\,\frac{1}{2}\,\hat{c}^\dagger F_K^\dagger \hat{c}^\dagger\,\frac{1}{2}\,\hat{c}\,F_K\,\hat{c}. \tag{20}$$

The one-body term includes the single-particle energy with matrix e, the fermion chemical potential μ_F and any single-particle interaction with external fields included in the matrix $\mathcal{M}$

$$h_0 = e - \mu_F + \mathcal{M}. \tag{21}$$

The matrices F_K are the form factors of the potential, normalized according to

$$\mathrm{tr}(F_{K_1}^\dagger F_{K_2}) = 2\,\Omega\delta_{K_1K_2}. \tag{22}$$

In order to evaluate Z_C I divide the inverse temperature in N_0 intervals of size τ

$$T = \frac{1}{N_0\tau}. \tag{23}$$

Then as shown in Appendix D the Euclidean effective action is

$$\begin{aligned} S_{\mathit{eff}}(\beta^*, \beta) = \tau \sum_t \frac{1}{2}\mathrm{tr}\left\{\frac{1}{\tau}\ln\left[\mathbb{1} + \tau\, R\,\mathcal{B}^\dagger \nabla_t \mathcal{B}\right]\right. \\ +2R\,\mathcal{B}^\dagger h\,\mathcal{B} - \sum_K g_K\left[(R-1)\,F_K^\dagger F_K\right. \\ \left.\left.+(R\mathcal{B}^\dagger F_K^\dagger)\,\frac{1}{2}\mathrm{tr}(RF_K\mathcal{B}) - R\mathcal{B}^\dagger F_K^\dagger R F_K \mathcal{B}\right]\right\} \end{aligned} \tag{24}$$

where

$$h = h_0 - \sum_K g_K F_K^\dagger F_K \tag{25}$$

$$\mathcal{B} = \frac{1}{\sqrt{\Omega}}\sum_J \beta_J B_J^\dagger = \frac{1}{\sqrt{\Omega}}\beta\cdot B^\dagger \tag{26}$$

$$R = \left(\mathbb{1} + \mathcal{B}^\dagger\mathcal{B}\right)^{-1} \tag{27}$$

Notice in the third line a trace inside the trace. *The variables β^*, β are always understood at times $t, t-1$ respectively.* Explicitly, for instance

$$R_t = \left(\mathbb{1} + \mathcal{B}_t^\dagger \mathcal{B}_{t-1}\right)^{-1}. \tag{28}$$

S_{eff} has a global $U(1)$ symmetry which implies boson conservation.

The fermionic interactions with external fields appear in the bosonic terms which involve the matrix $\mathcal{M}$ (appearing in h).

The dynamical problem of the interacting (composite) bosons can be solved within the path integral formalism. Part of this problem is the determination of the structure matrices B_J which can be done by a variational calculation. The present approach then shares two important features with variational methods: the restriction of the fermion space to a subspace, the space of the composites, and the variational determination of the structure functions. But unlike standard variational methods excited states are treated at the same time and on the same footing as the ground state.

Before showing a concrete way of application, I want to list some characteristic features of the present formalism:

(1) It can be used with interactions quartic as well quadratic in the fermionic fields. The latter ones include electromagnetic and phonon interactions in the nonrelativistic domain, and renormalizable relativistic field theories in 3+1 dimensions [23,24]. In the application to these theories the main change is the replacement of the exponential of the Hamiltonian by the transfer matrix.
(2) It can be extended to cases in which not all the fermions bosonize, like odd atomic nuclei, systems of fermions and composite bosons in quasichemical equilibrium, and relativistic field theories at finite fermion density. This latter case has already been studied for QCD at finite temperature and baryon density [24], and one can see that the procedure adopted can easily be applied to nonrelativistic systems as well.
(3) It allows the treatment of different coexisting composite bosons, for instance Cooper and molecular pairs. In such a case the equations for the structure functions of the composites should have more solutions, possibly with the same quantum numbers.
(4) It provides the structure functions of the composites, and therefore their effective coupling to other fields, like the electromagnetic field.
(5) It allows the introduction of composites with quantum numbers different from those of the form factors of the potential. An example is given in Section 6: with a purely pairing interaction in atomic nuclei, one needs also bosons with angular momentum different from zero.

4 The effective boson Hamiltonian

As already said the derivation of the boson Hamiltonian is not straightforward, because S_{eff} differs in many respects from the action of elementary bosons shown in equation (8). I notice that in S_{eff}

(i) the time derivative term is not canonical;
(ii) the coupling of the chemical potential (appearing in h) is also noncanonical;
(iii) there are non polynomial interactions because of the R-function. This function becomes singular, as it will become clear in the sequel, when the number of bosons is of order Ω, reflecting the Pauli principle.

Let us start by examining the features of compositeness when the number of bosons is much smaller than Ω. Since the expectation value of $\beta^*\cdot\beta$ is of the order of the number of bosons, in this case we can perform an expansion of logarithm and R-function in inverse powers of Ω. From the logarithm I get

$$\frac{1}{2\tau}\,\mathrm{tr}\ln\left[\mathbb{1}+\tau R\mathcal{B}^\dagger\nabla_t\mathcal{B}\right]=\frac{1}{2}\mathrm{tr}\left(\mathcal{B}^\dagger\nabla_t\mathcal{B}\right)-\frac{1}{4}\mathrm{tr}\left[\mathcal{B}^\dagger\mathcal{B}\,\mathcal{B}^\dagger\nabla_t\mathcal{B}\right]+\ldots \quad (29)$$

The first term can be made canonical by normalizing the boson form factors according to

$$\mathrm{tr}(B_J^\dagger\,B_K)=2\Omega\,\delta_{J,K}. \quad (30)$$

The other terms are then of order Ω^{-1}. Notice that the diagonal condition is only a matter of normalization, but the off diagonal one must be compatible with the dynamics, and a redefinition of the β's can be necessary to get it.

Expanding the R-function I get the following couplings of the fermion chemical potential

$$\mu_F\,\mathrm{tr}\left[\mathcal{B}_t^\dagger\mathcal{B}_{t-1}-\frac{1}{2}(\mathcal{B}_t^\dagger\mathcal{B}_{t-1})^2+\ldots\right]. \quad (31)$$

Only the first term is canonical. However the anomalous couplings can be eliminated to order Ω^{-1} by a redefinition of the chemical potential [21], so that in the case of a small number of bosons the Hamiltonian can be derived without difficulty.

But when the number of bosons is of order Ω, an expansion of logarithm and R-function can be performed only after an appropriate subtraction, as explained in Section 5. I assume, and I will verify later, that after such a subtraction the anomalous time derivative terms be of order Ω^{-1}. Then Z_C can be written in terms of an auxiliary Hamiltonian H' as a trace in a boson space

$$Z_C=\mathrm{tr}\exp\left(-\frac{1}{T}H'\right). \quad (32)$$

H' is obtained from S_{eff} by omitting the time derivative term, and replacing the variables β^*,β by corresponding creation-annihilation operators $\hat b^\dagger,\hat b$. These satisfy canonical commutation relations and should not be confused with the corresponding operators $\hat B^\dagger,\hat B$ for the composites

$$H'=:\frac{1}{2}\mathrm{tr}\Bigg\{2\hat R\,\hat{\mathcal{B}}^\dagger h\;\hat{\mathcal{B}}-\sum_K g_K\left[\left(\hat R-1\right)\,F_K^\dagger F_K+\hat R\,\hat{\mathcal{B}}^\dagger F_K^\dagger\frac{1}{2}\mathrm{tr}(\hat R\,F_K\hat{\mathcal{B}})\;-\hat R\,\hat{\mathcal{B}}^\dagger F_K^\dagger\hat R\,F_K\hat{\mathcal{B}}\right]\Bigg\}. \quad (33)$$

The colons denote normal ordering and

$$\hat{\mathcal{B}}=\frac{1}{\sqrt{\Omega}}\,\hat b\cdot B^\dagger$$
$$\hat R=\left[\mathbb{1}+\frac{1}{\Omega}\,\hat b^\dagger\cdot B\;\hat b\cdot B^\dagger\right]^{-1}. \quad (34)$$

From H' I will derive in the next section the boson Hamiltonian H_B, equation (55). I must notice that the case of a number of bosons much smaller than Ω cannot be retrieved from the above equations. The reason is that in the former case $\mu_F=O(\Omega)$, while $\mu_F=O(\Omega^0)$ in the case $n\sim\Omega$.

5 The Ω^{-1} expansion

A rather general way to perform the subtraction necessary for the Ω^{-1} expansion is to write the operator $\hat R$ in the form

$$\hat R=(\mathbb{1}+\hat\eta)^{-1}\,\Gamma \quad (35)$$

where

$$\hat{\eta} = \frac{1}{\Omega}\Gamma \sum_{K_1K_2} \left(\hat{b}^\dagger_{K_1}\hat{b}_{K_2} - r^2_{K_1K_2}\right) B_{K_1}B^\dagger_{K_2}$$
$$\Gamma = \left[\mathbb{1} + \frac{1}{\Omega}\sum_{K_1K_2} r^2_{K_1K_2} B_{K_1}B^\dagger_{K_2}\right]^{-1}. \quad (36)$$

Notice that $\hat{R}$ can be expanded with respect to $\hat{\eta}$, while Γ, which however does not contain creation-annihilation operators, must be treated exactly. The parameters $r^2_{K_1K_2}$ are related to the expectation values $\langle\hat{b}^\dagger_{K_1}\hat{b}_{K_2}\rangle$. It is important to observe that such expectation values do not break boson (and therefore fermion) number conservation, but will in general break other symmetries, like rotational invariance in deformed atomic nuclei. Their determination allows therefore to study thermodynamic or quantum phase transitions like breaking of rotational symmetry by axial or triaxial shapes in atomic nuclei [27].

For the sake of simplicity in [21] I restricted myself to cases in which the structure functions are almost constant, which justifies a subtraction independent of them. The formalism was then tested in the case of the so called pairing model [25]. The ground state energy was exactly reproduced, but the spectrum of excitations was not studied because the boson Hamiltonian contains couplings of all the bosons among themselves and was not solved. As we will see such couplings are an artifact due to the inadequacy of that subtraction.

5.1 Subtraction in the presence of an s-condensate

In the present paper I remove the restriction that the form factors should be almost constant, and consider the case in which a condensation occurs in a single quantum mode, called the s mode, whose quantum numbers will be denoted by "zero"

$$r^2_{K_1K_2} = \delta_{K_1,0}\,\delta_{K_2,0}\,r^2. \quad (37)$$

By condensation of the s-boson I understand that the occupation number of this mode is of order Ω. The terms appearing in $\hat{\eta}$ can then be classified according to

$$(\hat{b}^\dagger_0\hat{b}_0 - r^2)B^\dagger_0 B_0 + \sum_{K_1,K_2\neq 0}\hat{b}^\dagger_{K_1}\hat{b}_{K_2}\,B_{K_1}\,B^\dagger_{K_2} \;\sim\; \Omega^0$$
$$\hat{b}^\dagger_0\,B_0\sum_{K\neq 0}\hat{b}_K\,B^\dagger_K \;\sim\; \sqrt{\Omega}, \quad (38)$$

provided the sum of the occupation numbers of noncondensed modes is much smaller than Ω and the structure matrices are of order Ω^0. Under these conditions one subtraction is sufficient, otherwise more subtractions are needed. Expansion of time derivative terms in S_{eff} according to this classification, neglecting contributions of order Ω^{-1} gives

$$\frac{1}{2\tau}\mathrm{tr}\left[\ln\left(1+\mathcal{B}^\dagger_t\mathcal{B}_t\right) - \ln\left(1+\mathcal{B}^\dagger_t\mathcal{B}_{t-1}\right)\right] = \frac{1}{2\Omega}\mathrm{tr}\,C_{00}\,\beta^*_0\nabla_t\beta_0 + \sum_{K_1K_2}\frac{1}{2\Omega}\mathrm{tr}\,[C_{K_1K_2} - \frac{1}{\Omega}\beta^*_0\beta_0\,C_{K_10}C_{0K_2}\Big]\beta^*_{K_1}\nabla_t\beta_{K_2} \quad (39)$$

where

$$C_{K_1K_2} = \Gamma\,B_{K_1}B^\dagger_{K_2}. \quad (40)$$

In the derivation of the above equation I assumed $\mathrm{tr}\,C_{K0} = 0$, for $K \neq 0$ and I disregarded the fluctuations of the product $\beta^*_0\beta^*_0$, whose contribution is of order Ω^{-1}, so that terms of the form

$$\frac{1}{2\Omega^2}\mathrm{tr}(C_{0K_1}C_{0K_2})\beta^*_0\beta^*_0\nabla_t(\beta_{K_1}\beta_{K_2}) \quad (41)$$

are total time derivatives and do not contribute to the action. Then replacing $\beta^*_0\beta_0$ by n, the number of bosons, the temporal terms become canonical if I impose the normalizations

$$\frac{1}{2\Omega}\mathrm{tr}\,C_{00} = 1,$$
$$\frac{1}{2\Omega}\mathrm{tr}\left[C_{K_1K_2} - \frac{n}{\Omega}C_{K_10}C_{0K_2}\right] = \delta_{K_1K_2},$$
$$K_1, K_2 \neq 0. \quad (42)$$

It is worth while noticing that the ground state wave function of free fermions is included in the class of states which are assumed to dominate the partition function, so that the variational evaluation of the structure function will tell whether an actual condensation will occur or not. Indeed if I choose

$$(B_0B^\dagger_0)_{m_1m_2} = \xi\,\delta_{m_1m_2}\theta(m_F - m_1) \quad (43)$$

where θ is the step function and m_F are the fermion quantum numbers at the Fermi surface, for large ξ the normalization gives

$$r^2 = \frac{1}{2}n_F, \quad (44)$$

and all the coherent states are dominated by the term with n_F fermions

$$|\beta\rangle \sim (\beta\,\xi)^{\frac{1}{2}n_F}\prod_{m=1}^{m_F} c^\dagger_m. \quad (45)$$

5.2 Ω^{-1} expansion in the presence of an s condensate

To proceed with the expansion of the boson Hamiltonian it is necessary to know how the coupling constants g_K scale with Ω. For infinite systems, since $\Omega \to \infty$, to get finite energies we must require $g_K \sim \Omega^{-1}$. Such a behavior is also acceptable for many finite systems and I assume it in the following. As a consequence the fermion chemical potential μ_F is of order Ω^0.

At this point it is convenient to introduce the notations

$$\nu = \frac{r^2}{\Omega}, \quad \hat{n}_0 = \hat{b}_0^\dagger \hat{b}_0. \tag{46}$$

Neglecting contributions of order Ω^{-1} I get

$$H' \sim E_C + : \Bigg\{ \mathcal{E}_0\, \hat{n}_0 + \frac{1}{2} \sum_{K_1, K_2 \neq 0} \Big[\Big(\mathcal{E}_{K_1 K_2}\, \hat{b}_{K_1}^\dagger \hat{b}_{K_2} + \frac{1}{\Omega} \mathcal{V}_{K_1, K_2}\, \hat{b}_{K_1}^\dagger \hat{b}_{K_2}^\dagger \hat{b}_0 \hat{b}_0 \Big) + H.c. \Big] \Bigg\} :, \tag{47}$$

where E_C is a c-number and $H.c.$ represents the Hermitian conjugate of the operator in the round brackets. The coefficients $\mathcal{E}$ and $\mathcal{V}$ are functions of the operator $\hat{n}_0$, which explains the presence of normal ordering.

In many cases the double sum over K_1, K_2 reduces to a single sum, K_2 resulting conjugate to K_1 according to a one-to-one correspondence $K_2 = \tilde{K}_1$

$$H' \sim E_C + : \Bigg\{ \mathcal{E}_0\, \hat{n}_0 + \sum_{K \neq 0} \Big[\mathcal{E}_K \hat{b}_K^\dagger \hat{b}_K + \frac{1}{2\,\Omega} \mathcal{V}_K \hat{b}_K^\dagger \hat{b}_{\tilde{K}}^\dagger \hat{b}_0 \hat{b}_0 + H.c. \Big] \Bigg\} : . \tag{48}$$

From their general expression, reported in Appendix E, we see that $E_C \sim \Omega$ while the operators $\mathcal{E}_K, \mathcal{V}_K$ take values of order Ω^0.

H' has the form of the Bogoliubov model of superfluidity, with an important qualification to be discussed below. Because of the absence of terms involving three or four operators of bosons out of the condensate, H' can be approximately (see below) diagonalized introducing the phonon operators

$$\hat{A}_K = \frac{1}{\sqrt{n}} \hat{b}_K \hat{b}_0^\dagger, \quad \hat{A}_K^\dagger = \frac{1}{\sqrt{n}} \hat{b}_K^\dagger \hat{b}_0. \tag{49}$$

In terms of these operators

$$H' = E_C\ + \delta E_C + : \mathcal{E}_0 \hat{n}_0 : + \sum_{K \neq 0} E_K \hat{A}_K^\dagger \hat{A}_K \tag{50}$$

where

$$\delta E_C = \frac{1}{2} \sum_{K \neq 0} (E_K - \mathcal{E}_K)$$

$$E_K = \sqrt{\mathcal{E}_K^2 - \Big(\frac{n}{\Omega}\, \mathcal{V}_K \Big)^2}. \tag{51}$$

The average number of noncondensed bosons in the ground state is [28]

$$\langle \sum_{K \neq 0} \hat{b}_K^\dagger \hat{b}_K \rangle = \frac{1}{2} \sum_{K \neq 0} \left(\frac{\mathcal{E}_K}{E_K} - 1 \right). \tag{52}$$

Approximating the commutation relations of the phonon operators by canonical ones introduces errors of order Ω^{-1} only if this expectation value is much smaller than Ω. This condition is also necessary for the classification of equation (38) to hold. Now even though the operators $\mathcal{E}_K$ and $\mathcal{V}_K$ take values of order Ω^0, so that also the phonon energies E_K are of this order, it is well possible that $\langle \sum_{K \neq 0} \hat{b}_K^\dagger \hat{b}_K \rangle = O(\Omega)$. *In such a case a unique subtraction is not sufficient and farther subtractions are necessary.*

Now the qualification mentioned above. As reminded in the Introduction in his reformulation of the theory of superconductivity starting from the Fröhlich Hamiltonian [4] of electrons interacting with lattice phonons, Bogoliubov found an effective Hamiltonian essentially equal to that he studied previously for superfluid bosonic systems as far as the bosonic excitations are concerned. The coefficients $\mathcal{E}_K$ and $\mathcal{V}_K$ of this Hamiltonian are such [28] that the excitation spectrum is

$$E_K \sim |K|, \quad |K| \to 0. \tag{53}$$

As a consequence $\langle \sum_{K \neq 0} \hat{b}_K^\dagger \hat{b}_K \rangle, \delta E_C = O(\Omega)$. Such a result cannot hold for the BCS model, because the expression of the ground state energy density derived by BCS is exact [5].

5.3 Determination of subtraction parameter and fermion chemical potential

In the determination of the subtraction parameters I meet with a subtlety. H' commutes with the boson number operator, so I can select sectors with a given number of bosons. But I am not guaranteed that these bosons carry fermion number 2, because of the noncanonical coupling of the chemical potential. This fundamental property can be enforced just exploiting the parameters introduced by the subtractions. Indeed, denoting by $E_0'(n)$ the lowest eigenvalue of H' in the sector of n bosons, I require that $E_0'(n)$ be the lowest eigenvalue for $n = \frac{1}{2}\, n_F$

$$\frac{\partial}{\partial n} E_0'(n) = 0, \ \text{ for } n = \frac{1}{2} n_F. \tag{54}$$

The above equation, together with the condition 2 on the fermion number, determines one of the parameters r and the fermion chemical potential μ_F as functions of the number of bosons, $\overline{r} = \overline{r}(n), \overline{\mu}_F = \overline{\mu}_F(n)$, ensuring that these bosons carry fermion number 2. The boson Hamiltonian in the sector of n bosons is finally

$$H_B(n) = H'(\overline{r}, \overline{\mu}_F) + 2\, \overline{\mu}_F\, n. \tag{55}$$

It depends on n explicitly and through the dependence on n of $\overline{r}, \overline{\mu}_F$. Therefore also the matrices B_J, the form factors of the bosons, will depend on n, namely on the number of fermions.

Notice that H' provides a mapping of the fermion interactions with external fields

$$c^\dagger \mathcal{M} c \to : \frac{1}{2} \mathrm{tr} \left\{ \hat{\Gamma} \left[\hat{\mathcal{B}}^\dagger \mathcal{M} \hat{\mathcal{B}} + \frac{1}{2} [\hat{\mathcal{B}}^\dagger \hat{\mathcal{B}}, \mathcal{M}]_+ \right] \right\} : . \tag{56}$$

The ground state energy of the auxiliary Hamiltonian in a state of n bosons is

$$E_0'(n) = E_C + \delta E_C + \langle n,o| : \mathcal{E}_0(\hat{n}_0)\,\hat{n}_0 : |n,o\rangle. \quad (57)$$

Taking normal ordering into account

$$\langle n,o| : \mathcal{E}_0(\hat{n}_0)\,\hat{n}_0 : |n,o\rangle = \overline{\mathcal{E}}_0(n_0)\,n_0 \quad (58)$$

where

$$\overline{\mathcal{E}}_0(n_0) = \mathcal{E}_0(n_0) + 2n_0 g_0 D_{00} D_{0000}. \quad (59)$$

n_0 is the number of condensed bosons in a state of n bosons and the quantities $D_{K_1K_2\ldots}$ are defined in Appendix B. The constraints 2 and 54 for subtraction parameter and chemical potential are

$$\frac{\partial E_0'(n)}{\partial \mu_F} = -2\left\{ r^2 \frac{1}{2\Omega}\mathrm{tr}\, C_{00} + \left(n_0 - r^2\right)\frac{1}{2\Omega}\mathrm{tr}\left[\Gamma C_{00}\right]\right\} + \frac{\partial \delta E_C}{\partial \mu_F} + 2n = 0$$

$$\frac{\partial E_0'(n)}{\partial n} = \frac{\partial}{\partial n}\left[\overline{\mathcal{E}}_0(n_0)\,n_0\right] + \frac{\partial \delta E_C}{\partial n} = 0. \quad (60)$$

I have two equations and three unknowns: μ_F, r^2, n_0. But I remind that the Ω^{-1} expansion holds under the assumption

$$n - n_0 = O(\Omega^0), \quad (61)$$

otherwise further subtractions are necessary. I assume, and I will discuss later when this assumption holds true

$$\frac{\partial}{\partial \mu_F}\delta E_C = O(\Omega^0), \quad \frac{\partial}{\partial n}\delta E_C = O(\Omega^{-1}). \quad (62)$$

Then the first equation with the normalization condition (42) gives

$$\overline{r}^2 = n + O(\Omega^0), \quad (63)$$

and using this result in the second equation I get

$$\frac{1}{2\Omega}\mathrm{tr}\left[\Gamma^2 B_0 (e-\mu_F) B_0^\dagger\right] + 2\nu\,\Omega g_0 D_{00} D_{0000} - \Omega g_0 D_{00}^2 = O(\Omega^{-1}) \quad (64)$$

which determines $\overline{\mu}_F$.

I can finally write the ground state eigenvalue of H_B

$$E_0(n) = E_0'(n) + 2n\overline{\mu}_F = \left\{ \frac{1}{\Omega}\mathrm{tr}\left[\Gamma B_0\, e B_0^\dagger\right] - \Omega g_0 D_{00}^2 \right\} n + O(\Omega^0). \quad (65)$$

To evaluate the excitation energies in a state with f phonons I need the expectation value

$$\langle n,f| : \mathcal{E}_0(\hat{n}_0)\,\hat{n}_0 : |n,f\rangle = \overline{\mathcal{E}}_0(n_0 - f)(n_0 - f) \sim \overline{\mathcal{E}}_0(n_0) - \frac{\partial}{\partial n_0}\left[\overline{\mathcal{E}}_0(n_0)\,n_0\right] f. \quad (66)$$

According to equation (60) the coefficient of f is of order Ω^{-1}, so that to order Ω^0 this term does not depend on f.

The excitation energies of states containing f phonons are therefore

$$E_{K_1K_2\ldots K_f} = E_{K_1} + E_{K_2}\ldots + E_{K_f}. \quad (67)$$

6 Finite systems: the pairing model for nucleons in a single j-shell

I consider a system of nucleons in a single j-shell, in which case the only fermion quantum number is the third component of angular momentum m and the index of nilpotency Ω is equal to $j+1/2$. Composite bosons are labelled by the angular momentum L and its third component M: $K \equiv (L,M)$. The form factors of the potential are proportional to Clebsh-Gordan coefficients. With the normalization of equation (22) they are

$$(F_{LM})_{m_1,m_2} = \sqrt{2\Omega}\langle jm_1 jm_2|LM\rangle. \quad (68)$$

It is not necessary to solve the variational equations for the boson form factors, since because of rotational invariance they also must be proportional to the Clebsh-Gordan coefficients

$$B_{LM} = \alpha_L F_{LM}. \quad (69)$$

As a further simplification I restrict myself to the pairing model, namely I assume the single-particle energy to vanish and the interaction to be a pure pairing potential ($g_K = 0$ for $K \neq 0$). I will evaluate the excitation energies to order Ω^0 but the ground state energy only to order Ω since it was already evaluated to order Ω^0 in [21].

I can immediately find the normalizations

$$\alpha_0 = \frac{1}{\sqrt{1-\nu}}, \quad \alpha_L = \frac{1}{1-\nu}, \quad L \neq 0 \quad (70)$$

and evaluate the coefficient of H'. I find that $\tilde{L} = L, \tilde{M} = -M$, and

$$\begin{aligned} E_C &= 2h\Omega\nu^2 + \Omega g_0\,\nu \\ \mathcal{E}_0 &= \left[2h - g_0(\Omega + 2 + 2\nu\right](1-\nu) \\ \mathcal{E}_{LM} &= \left[2h(1-2\rho_0) + 4\Omega g_0\rho_0(1-\rho_0)\right] \\ \mathcal{V}_{LM} &= (-1)^M 2\left[-2h + \Omega g_0(1-2\rho_0)\right]. \end{aligned} \quad (71)$$

where $\rho_0 = n_0/\Omega$.

Equations (63) and (64) give

$$\begin{aligned} \overline{r}^2 &= n + O(\Omega^0) \\ \overline{\mu}_F &= -\frac{1}{2}g_0\,(\Omega - n) + O(\Omega^{-1}). \end{aligned} \quad (72)$$

With these values

$$\mathcal{V}_{LM} = O(\Omega^{-1}), \quad \mathcal{E}_{LM} = \Omega g_0 + O(\Omega^{-1}). \quad (73)$$

Since the number of L-modes is of order Ω^2, namely $\delta E_C = O(\Omega^0)$, the effective boson Hamiltonian in the sector of $2n$ fermions is

$$H_B(n) = \left[-\Omega g_0 n + g_0 n^2 + O(\Omega^0)\right] + \left[\Omega g_0 \sum_{LM}\hat{b}_{LM}^\dagger \hat{b}_{LM} + O(\Omega^{-1})\right]. \quad (74)$$

This is the correct result [25] in the required approximation. It is to be noted that noncondensed bosons have all the same energy $E_{LM} = \Omega g_0$, so that the spectrum depends only on their total number, and not on their distribution in different L-states. *This property makes consistent the truncation of the boson space to a few modes*, and as a consequence we can restrict the boson model space to only one of them, the d-boson in the Interacting Boson Model.

The decoupling of different noncondensed bosons is achieved by means of the subtraction of equation (37). With a subtraction independent of the structure functions all the modes are coupled, so that the above properties might have been found only after solution of the resulting boson Hamiltonian.

7 Infinite systems: superconductors and the BCS model

I consider an infinite system of fermions whose quantum numbers are spin and momentum, $m \equiv (s,p)$, and I parametrize the potential form factors according to

$$(F_K)_{s_1p_1s_2p_2} = \epsilon_{s_1,s_2}\delta_{p_1+p_2,K}\, f_{p_1-\frac{1}{2}K}(K). \tag{75}$$

For simplicity I restrict myself to bosons of spin zero and momentum K and parametrize their form factors according to

$$(B_K)_{s_1p_1,s_2p_2} = \epsilon_{s_1,s_2}\delta_{p_1+p_2,K}\, \phi_{p_1-\frac{1}{2}K}(K). \tag{76}$$

Since form factor and structure function are fully antisymmetric

$$f_q(K) = f_{-q}(K),\ \ \phi_q(K) = \phi_{-q}(K). \tag{77}$$

The structure functions are subject to the normalizations of equations (42)

$$\begin{aligned} \frac{1}{\Omega}\sum_p \Gamma_p\,\phi_p^2 &= 1 \\ \frac{1}{\Omega}\sum_p \Gamma_p^2\,\phi^2_{p-\frac{1}{2}K}(K) &= 1,\ \ K\neq 0 \end{aligned} \tag{78}$$

where

$$\phi_p = \phi_p(0),\ \ \Gamma_p = \left(1+\nu\,\phi_p^2\right)^{-1}. \tag{79}$$

For infinite systems neglecting terms of order Ω^0 in the evaluation of the ground state energy density does not produce any error in thermodynamic limit. So I will neglect such terms. Therefore the ground state energy is

$$\begin{aligned} E_0 &= \delta E_C + \left\{\frac{1}{\Omega}\mathrm{tr}\left[\Gamma B_0\, e\, B_0^\dagger\right] - \Omega g_0 D_{00}^2\right\} n \\ &= \delta E_C + \left\{\sum_p 2\, e_p\,\Gamma_p\phi_p^2 - g_0\left[\sum_p \Gamma_p\phi_p f_p\right]^2\right\}\frac{n}{\Omega}. \end{aligned} \tag{80}$$

The expressions of $\delta E_C, \mathcal{E}_K$ and $\mathcal{V}_K$ can be easily evaluated inserting the definitions of potential form factors and boson structure functions in the equations reported in Appendix B.

7.1 BCS model

For a general potential $\delta E_C \neq O(\Omega^0)$, and as already said consistency of the Ω^{-1} expansion requires further subtractions. But a great simplification must arise for a pure pairing interaction, $g_K = 0$ for $K \neq 0$, because then as reminded previously the contribution of δE_C to the energy per particle must vanish in the thermodynamic limit, or in other words δE_C must be of order Ω^0. Therefore I assume $\delta E_C = O(\Omega^0)$ and I will verify a posteriori that this is true. I also assume ϕ_p to be real and I minimize the ground state energy imposing the normalization constraint by the Lagrange multiplier λ

$$\frac{\partial}{\phi_q}\left\{\sum_p 2(e_p-\lambda)\Gamma_p\phi_p^2 - g_0\left[\sum_p \Gamma_p\phi_p f_p\right]^2\right\} = 0. \tag{81}$$

I thus get the gap equation

$$(e_p-\lambda)\,\sqrt{\nu}\phi_p = \frac{1}{2}\,\triangle\,(1-\nu\,\phi_p^2) f_p \tag{82}$$

where

$$\triangle = g_0\sqrt{\nu}\sum_p \Gamma_p\,\phi_p\,f_p \tag{83}$$

is the gap function. *These are exactly the results of the quasi-chemical equilibrium theory, equation IV(1.17) of reference [29].* In this connection I remind that this theory, unlike the BCS theory, is fermion number conserving like the present approach.

Following [29] I then find $\lambda = e_{p_F}, p_F$ being the Fermi momentum of the system. Next I evaluate the coefficient of the quartic boson term

$$\begin{aligned} \mathcal{V}_K = -\frac{2}{\Omega}\sum_p \Big\{ & \Gamma_p^2\,\Gamma_{p-K}\,[2(e_p-\overline{\mu}_F)\phi_p \\ & -\frac{1}{\sqrt{\nu}}\left(1-\nu\phi_p^2\right)\triangle\, f_p\Big]\phi_{p-K} \\ & \times\phi_{p-\frac{1}{2}K}(K)\phi_{p-\frac{1}{2}K}(-K)\Big\} + O(\Omega^{-1}). \end{aligned} \tag{84}$$

In the above equation there appears the fermion chemical potential $\overline{\mu}_F$ which is determined by equation (64). Using the gap equation I find $\overline{\mu}_F = \lambda$. Then using again the gap equation in the expression of $\mathcal{V}_K$ I find that $\mathcal{V}_K = O(\Omega^{-1})$ and therefore δE_C is at most $O(\Omega^0)$. *For a pure pairing interaction the contribution to the ground state energy per particle of noncondensed pairs vanishes, in agreement with [5], and my assumption about δE_C is justified.*

To get analytic expressions I assume e_p to be the free fermion energy and the form factor of the potential to have the schematic form of BCS

$$e_p = \frac{1}{2m}p^2,\ \ f_p = \theta(2m\omega - |p^2 - p_F^2|). \tag{85}$$

θ is the step function and ω a cutoff energy (identified with the Debye cutoff energy of lattice waves in metal superconductivity).

The solution of the gap equation is [29]

$$\phi_p = \frac{1}{\nu}\left(\sqrt{1+\xi_p^2} - \xi_p\right) \tag{86}$$

where

$$\xi_p = \frac{1}{\triangle}\frac{1}{2\,m}(p^2 - p_F^2). \tag{87}$$

8 Summary and outlook

I extended the original formalism for boson dominance based on coherent composite states. The extension amounts to perform a subtraction proportional to the structure function of the condensed boson. The results obtained have a validity restricted only by the assumption that condensation occurs in a singlet state and that the coupling constants scale according to $g_K \sim \Omega^{-1}$. In particular, since fermion number is conserved, they hold for finite systems. Therefore the present bosonization method can find applications not only in the presence of important nonpairing interactions, but also to account for finiteness effects in atomic nuclei and small metallic grains.

The general expression of the bosonic Hamiltonian contains a classical part δE_C which couples the form factors of all the bosons. The theory becomes much simpler for pure pairing interactions, because δE_C is negligible, and I tested it on two paradigmatic models.

As an example of finite systems I considered the pairing model of nucleons in a single j-shell and I reproduced its bosonic spectrum in a form useful to understand and justify the Interacting Boson Model. *Indeed I find that a boson space made of the s- and d-bosons is sufficient to reproduce the spectrum, which is not changed by the addition of other bosons.* For the fermionic part of the spectrum one must use the technique developed in [24]. Needless to say the bosonization of the pairing model has been obtained in many ways, but I emphasize that the present formulation holds in a framework valid for arbitrary fermion-fermion interactions.

I then applied this method to the BCS model of superconductivity reproducing exactly its ground state properties in the form of the quasi-chemical equilibrium theory.

Two important issues are left for future work. The first one concerns the determination of structure functions for a general interaction and noncondensed bosons. I keep in mind in this connection the possibility of different structure functions with the same quantum number K. They might for instance be associated with intruders or coexisting molecular and Cooper pairs.

The second one concerns phonons, namely neutral bosons describing polarization effects [30]. They must necessarily be included in many cases [31] and certainly in the presence of particle-hole terms in the fermion-fermion interaction.

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Appendix A: Basic formulae in Berezin integrals

The definition of the Berezin integral for a single Grassmann variable is

$$\int d\gamma(a\gamma + b) = a, \tag{A.1}$$

the generalization to many variables being obvious. For a change of variables

$$\gamma_i = \gamma_i(\gamma') \tag{A.2}$$

in a multiple integral we have

$$\int \prod_i (d\gamma_i) f(\gamma) = \left(\det \frac{\partial \gamma_h}{\partial \gamma'_k}\right)^{-1} \int \prod_i (d\gamma'_i) f(\gamma'). \tag{A.3}$$

Notice the appearance of the inverse of the jacobian, contrary to the case of ordinary variables.

Gaussian integrals can be evaluated exactly, like for ordinary variables. There are two types of such integrals

$$\int \prod_h (d\gamma_h^* d\gamma_h) \exp \sum_{ij} \gamma_i^* M_{ij} \gamma_j = \det M \tag{A.4}$$

$$\int \prod_h (d\gamma_h) \exp \sum_{ij} \frac{1}{2} \gamma_i A_{ij} \gamma_j = \mathrm{Pf}\, A \tag{A.5}$$

where $\mathrm{Pf}\, A$ is called the $pfaffian$ [32] of A. The following algebraic identity holds

$$(\mathrm{Pf}\, A)^2 = \det A. \tag{A.6}$$

Appendix B: Inner products of composite states

Let us consider the case of only one composite. To evaluate the inner product of coherent states I use the identity operator in the fermion Fock space

$$\mathcal{I} = \int d\gamma^* d\gamma \langle\gamma|\gamma\rangle^{-1} |\gamma\rangle\langle\gamma| \tag{B.1}$$

where the γ^*, γ are Grassmann variables and $|\gamma\rangle$ coherent states [32]

$$|\gamma\rangle = \exp(-\gamma\, \hat{c}^\dagger)|\rangle. \tag{B.2}$$

I then have

$$\langle\beta_1|\beta\rangle = \langle\beta_1|\mathcal{I}|\beta\rangle = \int d\gamma^* d\gamma \, \exp(-\gamma^*\gamma) \langle\beta_1|\gamma\rangle\langle\gamma|\beta\rangle. \tag{B.3}$$

The matrix element $\langle\beta_1|\gamma\rangle$ can be evaluated using the defining property of coherent states

$$\hat{c}|\gamma\rangle = \gamma|\gamma\rangle \tag{B.4}$$

with the result

$$\langle\beta_1|\gamma\rangle = \exp\left(\frac{1}{2\sqrt{\Omega}}\beta_1^*\gamma B\gamma\right). \tag{B.5}$$

Therefore $\langle\beta_1|\beta\rangle$ becomes

$$\langle\beta_1|\beta\rangle = \int d\gamma^* d\gamma\, E(\gamma^*,\gamma,\beta_1^*,\beta), \tag{B.6}$$

where the function E is

$$E(\gamma^*,\gamma,\beta^*,\beta) = \exp\left(-\gamma^*\gamma + \frac{1}{2\sqrt{\Omega}}\beta^*\,\gamma\,B\,\gamma + \frac{1}{2\sqrt{\Omega}}\beta\,\gamma^* B^\dagger\gamma^*\right). \tag{B.7}$$

By the change of variables

$$\gamma' = \gamma^* - \frac{\sqrt{\Omega}}{\beta}(B^\dagger)^{-1}\gamma \tag{B.8}$$

the integral is factorized according to

$$\langle\beta_1|\beta\rangle = \int d\gamma' \exp\left(\frac{1}{2\sqrt{\Omega}}\gamma'\,\beta\,B^\dagger\,\gamma'\right) \times \int d\gamma \exp\left[\frac{1}{2}\gamma\left(\sqrt{\Omega}\left(\beta B^\dagger\right)^{-1} + \frac{1}{\sqrt{\Omega}}\beta_1^* B\right)\gamma\right].$$

The factors are of the form A.5, so that finally

$$\langle\beta_1|\beta\rangle = \left[\det\left(\frac{1}{\sqrt{\Omega}}\beta B^\dagger\right)\right]^{\frac{1}{2}}\left[\det\left(\sqrt{\Omega}(\beta B^\dagger)^{-1} + \frac{1}{\sqrt{\Omega}}\beta_1^* B\right)\right]^{\frac{1}{2}} = \det\left[\mathbb{1} + \frac{1}{\sqrt{\Omega}}\beta\,\beta_1^* B^\dagger B\right]^{\frac{1}{2}}. \tag{B.9}$$

Appendix C: The operator $\mathcal{P}$

In this Appendix I show that the operator $\mathcal{P}$ of equation (11) approximates $\mathcal{P}_C$ to leading order in an expansion in the inverse of the index of nilpotency.

For the sake of simplicity I consider the case of a unique composite. Then

$$\mathcal{P}_C = \sum_{n=0}^{\Omega}\frac{1}{\nu_n}|\left(\hat{B}^\dagger\right)^n|0\rangle\langle 0|\hat{B}^n| \tag{C.1}$$

where

$$\nu_n = \langle 0|\hat{B}^n|\left(\hat{B}^\dagger\right)^n|0\rangle. \tag{C.2}$$

I must then prove that

$$\langle 0|\hat{B}^m\,\mathcal{P}(\hat{B}^\dagger)^n|0\rangle \sim \langle 0|\hat{B}^m|(\hat{B}^\dagger)^n|0\rangle = \delta_{m,n}\nu_m. \tag{C.3}$$

These equations are generated by the following ones

$$\langle\beta'|\mathcal{P}|\beta''\rangle \sim \langle\beta'|\beta''\rangle \tag{C.4}$$

taking derivatives with respect to β'^*, β'' and setting these variables equal to zero. To simplify the formulae I adopt a slightly different definition of composites

$$\hat{B} = \frac{1}{2}\sum_{m_1 m_2}\hat{c}^\dagger_{m_1}B_{m_1 m_2}\hat{c}_{m_2}. \tag{C.5}$$

The right and left hand sides of Eq. (C.4) are

$$\langle\beta'|\mathcal{P}|\beta''\rangle = \int\frac{d\beta^* d\beta}{2\pi i}\exp\left[-\mathcal{E}(\beta^*,\beta,\beta'^*,\beta'')\right]$$
$$\langle\beta'|\beta''\rangle = \exp\left[\mathrm{Tr}\ln(1+\beta'^*\beta''BB^\dagger)\right] \tag{C.6}$$

where

$$\mathcal{E}(\beta^*,\beta,\beta'^*,\beta'') = \mathrm{Tr}\left[\ln(1+\beta^*\beta BB^\dagger) - \ln(1+\beta'^*\beta BB^\dagger) - \ln(1+\beta^*\beta''BB^\dagger)\right]. \tag{C.7}$$

I evaluate the integral by the saddle point method. The saddle point equations are

$$(\beta-\beta'')\mathrm{Tr}\frac{BB^\dagger}{(1+\beta^*\beta BB^\dagger)(1+\beta^*\beta''BB^\dagger)} = 0$$
$$(\beta^*-\beta'^*)\mathrm{Tr}\frac{BB^\dagger}{(1+\beta^*\beta BB^\dagger)(1+\beta'^*\beta BB^\dagger)} = 0 \tag{C.8}$$

with solutions

$$\overline{\beta} = \beta'', \quad \overline{\beta}^* = \beta'^*. \tag{C.9}$$

At the saddle point

$$\mathcal{E}(\overline{\beta}^*,\overline{\beta},\beta'^*,\beta'') = -\mathrm{Tr}\ln(1+\beta'^*\beta''BB^\dagger). \tag{C.10}$$

Moreover

$$\frac{\partial^2\mathcal{E}}{\partial\beta^*\partial\beta^*}|_{\overline{\beta}=\beta'',\overline{\beta}^*=\beta'^*} = \frac{\partial^2\mathcal{E}}{\partial\xi\partial\xi}|_{\overline{\beta}=\beta'',\overline{\beta}^*=\beta'^*} = 0$$
$$\frac{\partial^2\mathcal{E}}{\partial\beta^*\partial\beta}|_{\overline{\beta}=\beta'',\overline{\beta}^*=\beta'^*} = \mathrm{Tr}\frac{BB^\dagger}{(1+\beta'^*\beta''BB^\dagger)^2}. \tag{C.11}$$

In conclusion

$$\langle\beta'|\mathcal{P}|\beta"\rangle \sim \langle\beta'|\beta"\rangle\left[\mathrm{Tr}\frac{BB^\dagger}{(1+\beta'^*\beta''BB^\dagger)^2}\right]^{-1}. \tag{C.12}$$

The desired result follows if we assume

$$\mathrm{Tr}\left(B^\dagger B\right)^n \sim \Omega^{-n+1}. \tag{C.13}$$

It is then easy to prove also the idempotency property of projectors

$$\mathcal{P} \sim \mathcal{P}^2. \tag{C.14}$$

Appendix D: Derivation of the effective action

For the following manipulations we need the Hamiltonian in antinormal form

$$H = \frac{1}{2}\mathrm{tr}(h+h_0) - \hat{c}\, h^T \hat{c}^\dagger - \sum_K g_K \frac{1}{2}\hat{c} F_K \hat{c} \frac{1}{2}\hat{c}^\dagger F_K^\dagger \hat{c}^\dagger \quad \text{(D.1)}$$

where the upper script T means "transposed" and h was given in equation (25). Now we must evaluate the matrix element $\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle$. To this end we expand to first order in τ (which does not give any error in the final $\tau \to 0$ limit) and insert the operator $\mathcal{P}$ between annihilation and creation operators

$$\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \exp\left(-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right)\ \langle\beta_t|\mathcal{P}$$
$$-\hat{c}\, h^T \tau\, \mathcal{P}\hat{c}^\dagger \sum_k g_k \tau \frac{1}{2}\hat{c} F_K \hat{c}\, \mathcal{P} \frac{1}{2}\hat{c}^\dagger F_K^\dagger \hat{c}^\dagger |\beta_{t-1}\rangle. \quad \text{(D.2)}$$

Using the identity in the fermion Fock space we find

$$\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \int d\gamma^* d\gamma\, E(\gamma^*,\gamma,\beta_t^*,\beta_{t-1})$$
$$\times \exp\left(-\frac{1}{2}\mathrm{tr}(h+h_0)\tau - \gamma^* h\,\tau\gamma\right)$$
$$\times \exp\left(\sum_K g_K \tau \frac{1}{2}\gamma\, F_K\, \gamma \frac{1}{2}\gamma^* F_K^\dagger \gamma^*\right) \quad \text{(D.3)}$$

where the function $E(\gamma^*,\gamma,\beta^*,\beta)$ is defined in (B.7). By means of the Hubbard-Stratonovich transformation we can make the exponents quadratic in the Grassmann variables and evaluate the Berezin integral

$$\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \int \prod_K da_K^* da_K \exp(-a^*\cdot a)$$
$$\times \exp\left\{\frac{1}{2}\mathrm{tr}\ln\left[\mathbb{1} + \left(\mathcal{B}_t^* + \sum_{K_1}\sqrt{g_{K_1}\tau}\, a_{K_1}^* F_{K_1}\right)\right.\right.$$
$$\left.\left.\times R^{-1}\left(\mathcal{B}_{t-1} + \sum_{K_2}\sqrt{g_{K_2}\tau}\, a_{K_2}(F_{K_2})^\dagger\right)(R^T)^{-1}\right]\right\}$$
$$\times \det\rho\, \exp\left(-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right), \quad \text{(D.4)}$$

where

$$\rho = \mathbb{1} + h\,\tau. \quad \text{(D.5)}$$

Performing the integral over the auxiliary fields a_K^*, a_K we get

$$\langle\beta_t|\exp(-\tau H)|\beta_{t-1}\rangle = \int \prod_K da_K^* da_K \exp(-a^*\cdot a)$$
$$\times \exp\left\{\frac{1}{2}\mathrm{tr}\ln\left[\mathbb{1} + \left(\mathcal{B}_t^* + \sum_{K_1}\sqrt{g_{K_1}\tau}\, a_{K_1}^* F_{K_1}\right)\right.\right.$$
$$\left.\left.\times\rho^{-1}\left(\mathcal{B}_{t-1} + \sum_{K_2}\sqrt{g_{K_2}\tau}\, a_{K_2}(F_{K_2})^\dagger\right)(\rho^T)^{-1}\right]\right\}$$
$$\times \det\rho\, \exp\left(-\frac{1}{2}\mathrm{tr}(h+h_0)\tau\right), \quad \text{(D.6)}$$

The functional form of the composites partition function is

$$Z_C = \int \left[\frac{d\beta^* d\beta}{2\pi i}\right] \exp\left(-S_{\mathit{eff}}(\beta^*,\beta)\right) \quad \text{(D.7)}$$

where S_{eff} is given in equation (24).

Appendix E: General form of the coefficients in the auxiliary Hamiltonian

I will use the definitions

$$T_{K_1 K_2 \ldots K_{2l-1} K_{2l}} = \frac{1}{\Omega}\mathrm{tr}\left[C_{K_1 K_2}\ldots C_{K_{2l-3}K_{2l-2}} \times \Gamma B_{K_{2l-1}}\, h\, B_{K_{2l}}^\dagger\right]$$
$$D_{K_1 K_2 \ldots K_{2l-1} K_{2l}} = \frac{1}{2\Omega}\mathrm{tr}\left[C_{K_1 K_2}\ldots C_{K_{2l-3}K_{2l-2}} \times \Gamma B_{K_{2l-1}} F_{K_{2l}}^\dagger\right], \quad \text{(E.1)}$$

for l an arbitrary integer. Let me emphasize that only the last factor in T involves the single particle kinetic energy h, and only the last factor in D involves the form factors of the potential. *In the following I neglect the coupling to external fields and I assume that if only 2 indices K_1, K_2 are different from zero, they must be equal in the $\mathcal{E}$-terms.*

The classical energy is

$$E_C = \Omega\,\nu^2 T_{0000} + \sum_K \Omega g_K \nu \frac{1}{2\Omega}\mathrm{tr}\left[C_{00}F_K^\dagger F_K\right] \quad \text{(E.2)}$$

The s-boson energy is given by

$$\mathcal{E}_0 = T_{00} - \nu\, T_{0000} + \Omega g_0\left[-|D_{00}|^2 + 2\left(\frac{\hat{n}_0}{\Omega} - \nu\right)D_{00}D_{0000}\right] + \sum_K g_K \frac{1}{2\Omega}\mathrm{tr}\left[\Gamma B_0 F_K^\dagger \Gamma F_K B_0^\dagger\right]. \quad \text{(E.3)}$$

The terms $\mathcal{E}, \mathcal{V}$ are separated in their contributions from the kinetic and potential terms in the fermion Hamiltonian

$$\mathcal{E}_{K_1 K_2} = \mathcal{E}_{K_1 K_2}^{\mathrm{kin}} + \mathcal{E}_{K_1 K_2}^{\mathrm{pot}}$$
$$\mathcal{V}_{K_1, K_2} = \mathcal{V}_{K_1, K_2}^{\mathrm{kin}} + \mathcal{V}_{K_1, K_2}^{\mathrm{pot}} \quad \text{(E.4)}$$

whose expressions are

$$\mathcal{E}^{\rm kin}_{K_1,K_2} = T_{K_1K_2} - \nu\, T_{K_1K_200} - [T_{K_100K_2} + T_{0K_2K_10} - \nu(T_{0K_2K_1000} + T_{K_100K_200}]\,\frac{n}{\Omega}$$

$$\mathcal{E}^{\rm pot}_{K_1K_2} = 2\left\{ g_0 D_{00}\left[D_{K_2K_100} + D_{0K_1K_20} - \frac{\hat{n}_0}{\Omega}\,(D_{0K_1K_2000} + D_{K_200K_100})\right]^\dagger \hat{n}_0 + \sum_{K\neq 0} \hat{n}_0\, g_K \left[D_{K_1K} D^\dagger_{K_200K} - \frac{\hat{n}_0}{2\Omega}\left(D^\dagger_{0K_10K} D_{0K_20K} + D_{K_100K} D^\dagger_{K_200K}\right)\right]\right\}$$

$$\mathcal{V}^{\rm kin}_{K_1K_2} = -2\,[T_{K_10K_20} - \nu\, T_{K_10K_2000}]$$

$$\mathcal{V}^{\rm pot}_{K_1K_2} = 2\left\{ \Omega g_0 D_{00}\left[D_{K_20K_10} - \frac{\hat{n}_0}{\Omega}\,(D_{0K_10K_200} + D_{K_10K_2000})^\dagger\right] + \sum_{K\neq 0} \Omega g_K \left[D_{K_1K} D^\dagger_{0K_20K} - \frac{\hat{n}_0}{\Omega} D^\dagger_{0K_10K} D_{K_200K}\right]\right\}. \qquad \text{(E.5)}$$

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